An Introduction to the Logic of Partial Functions

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Outline

1. Partial Functions

2. Approaches
   - Changing the Notion of Equality
   - McCarthy's Conditional Operators
   - The Logic of Partial Functions

Partial Functions

- Partial function: A function whose application may lead to an undefined (non-denoting) term.
- Partial functions and operators arise frequently:
  - Division
  - Recursive function definitions
  - ...
- Classical logic does not handle undefined terms:
  - Terms must be denoting

Partial Functions Continued...

The *subp* Function

$$subp : \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}$$

$$subp(i, j) \triangleq \textbf{if } i = j \textbf{ then } 0 \textbf{ else } subp(i, j + 1) + 1$$

Property 1

$$\forall i, j \in \mathbb{Z} \cdot i \geq j \Rightarrow subp(i, j) = i - j$$

$\Rightarrow$	**true**	**false**
true	**true**	**false**
false	**true**	**true**

Approaches

- How do you handle undefined terms?
- John Harrison:
  - Four main approaches to handling undefined terms:
    - Give a value for input outside of the domain
    - Arbitrary value
    - Type error
    - Logic of partial terms
- We will now consider three approaches in more detail.

Changing the Notion of Equality

- Weak (strict) equality as we have seen is undefined if either operand is undefined.
- Alternative: Provide non-strict versions of equality which will denote a Boolean value even with undefined terms as operands.
- Weak equality

$=$	0	1	-	$\perp_{\mathbb{Z}}$
0	**true**	**false**	-	$\perp_{\mathbb{B}}$
1	**false**	**true**	-	$\perp_{\mathbb{B}}$
-	-	-	-	-
$\perp_{\mathbb{Z}}$	$\perp_{\mathbb{B}}$	$\perp_{\mathbb{B}}$	-	$\perp_{\mathbb{B}}$

Existential equality

$=_{\exists}$	0	1	-	$\perp_{\mathbb{Z}}$
0	**true**	**false**	-	**false**
1	**false**	**true**	-	**false**
-	-	-	-	-
$\perp_{\mathbb{Z}}$	**false**	**false**	-	**false**

Changing the Notion of Equality Continued...

Property 1

$$\forall i, j \in \mathbb{Z} \cdot i \geq j \;\Rightarrow\; subp(i, j) =_{\exists} i - j$$

- The non-strict equalities are not computable.
- Not just required for equality.
- Need to be aware of more than one notion of the operators in proofs:
    - Strict (computable) operators in function definitions
    - Non-strict operators to handle undefined terms

McCarthy's Conditional Operators

- Interpret the propositional operators as if they are defined by conditional expressions:
  - *p* **cor** *q* | **if** *p* **then** *true* **else** *q*
  - *p* **cimp** *q* | **if** *p* **then** *q* **else** *true*
  - ...
- Loss of commutativity (**cand** and **cor**), and contrapositive of **cimp**.
- Inevitable variable
- Alternative: Use alongside the standard logical operators:
  - Distribution is now complicated

McCarthy's Conditional Operators Continued...

- p **cimp** q | **if** p **then** q **else** *true*

Property 1

$\forall i, j \in \mathbb{Z} \cdot i \geq j$ **cimp** $subp(i, j) = i - j$

- p **cor** q | **if** p **then** *true* **else** q

Property 2

$\forall i, j \in \mathbb{Z} \cdot subp(i, j) = i - j$ **cor** $subp(j, i) = j - i$

The Logic of Partial Functions

- Vienna Development Method (VDM)
- A first-order predicate logic, which admits to undefined logical terms.
- Three-valued logic: *true*, *false*, and *undefined* ($\perp$).
- The truth tables are the strongest extension of their classical FOPC interpretations.

$\vee$	**true**	$\perp_{\mathbb{B}}$	**false**
true	**true**	**true**	**true**
$\perp_{\mathbb{B}}$	**true**	$\perp_{\mathbb{B}}$	$\perp_{\mathbb{B}}$
false	**true**	$\perp_{\mathbb{B}}$	**false**

$\Rightarrow$	**true**	$\perp_{\mathbb{B}}$	**false**
true	**true**	$\perp_{\mathbb{B}}$	**false**
$\perp_{\mathbb{B}}$	**true**	$\perp_{\mathbb{B}}$	$\perp_{\mathbb{B}}$
false	**true**	**true**	**true**

- Parallel evaluation of the operands.
- Return a result as soon as enough information is available.

The Logic of Partial Functions Continued...

- No law of the excluded middle ($e \vee \neg e$):
  - $subp(0, 5) = -5 \vee \neg(subp(0, 5) = -5)$
- Definedness operator (δ):
  - $\delta(e) = e \vee \neg e$

Property 1

$$\forall i, j \in \mathbb{Z} \cdot i \geq j \;\Rightarrow\; subp(i, j) = i - j$$

Property 2

$$\forall i, j \in \mathbb{Z} \cdot subp(i, j) = i - j \vee subp(j, i) = j - i$$

- Mechanisation

References

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Thank you.
Any Questions?